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LETTER TO THE EDITOR

Uncertainty product of composite signals

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Abstract

The well known uncertainty product of communication theory for a signal in the time domain and its Fourier transform in the frequency domain is studied for a 'composite signal', i.e. a 'pure' signal to which a time-delayed replica is added. This uncertainty product shows the appearance of local maxima and minima as a function of the time delay, leading to the following conjecture:

the uncertainty product of a non-Gaussian composite signal can be smaller than that of the 'pure' signal.

As an example this conjecture will be proven for the derivative of the Gaussian signal and for the Cauchy distribution. The effect on the uncertainty product of adding a delayed scaled replica of a signal to the original signal in the time domain leads to an important possibility for interpretation in the study of the reverberation phenomenon in echo-location signals of dolphins.

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1. Introduction

Consider a possibly complex-valued signal $s(t) \in \boldsymbol{L^2}(\mathbb{R})$; then its Fourier transform

$$\tilde{s}(f) = \int_{-\infty}^{\infty} \exp(-2\pi \mathrm{j} f t) s(t)\, \mathrm{d}t$$

is again square integrable over the real axis, $\tilde{s}(f) \in \boldsymbol{L^2}(\mathbb{R})$, and we have Parseval's equality

$$\|s\|^2 = \int_{-\infty}^{\infty} |s(t)|^2\, \mathrm{d}t = \int_{-\infty}^{\infty} |\tilde{s}(f)|^2\, \mathrm{d}f = \|\tilde{s}\|^2.$$

The functions $|s(t)|^2 / \int_{-\infty}^{\infty} |s(t)|^2\, \mathrm{d}t$ and $|\tilde{s}(f)|^2 / \int_{-\infty}^{\infty} |\tilde{s}(f)|^2\, \mathrm{d}f$ can be seen as densities in the time (respectively frequency) domain.

Introduce the mean time and frequency by

$$\bar{t} = \frac{\int_{-\infty}^{\infty} t|s(t)|^2\, \mathrm{d}t}{\int_{-\infty}^{\infty} |s(t)|^2\, \mathrm{d}t} \qquad \overline{f} = \frac{\int_{-\infty}^{\infty} f|\tilde{s}(f)|^2\, \mathrm{d}f}{\int_{-\infty}^{\infty} |\tilde{s}(f)|^2\, \mathrm{d}f} \tag{1}$$

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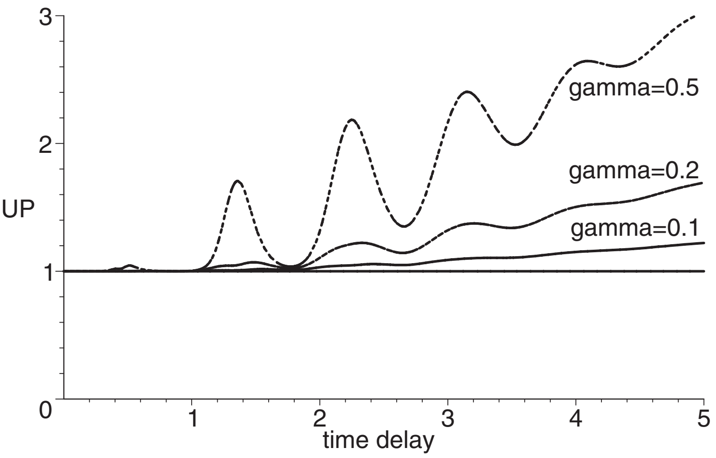

Figure 1. UP for the composite Gabor signal, $\omega = 7$.

and the variances by

$$\sigma_t^2 = \int_{-\infty}^{\infty} (t - \bar{t})^2 |s(t)|^2 \, dt \qquad \sigma_f^2 = \int_{-\infty}^{\infty} (f - \overline{f})^2 |\tilde{s}(f)|^2 \, df. \tag{2}$$

Relating time duration and frequency width as $\Delta t = 2\pi\sigma_t$ and $\Delta f = 2\sigma_f$ we then arrive at the following *uncertainty product* (UP for short):

$$\Delta t \Delta f = 4\pi \sigma_t \sigma_f. \tag{3}$$

The normalization used in (3) leads to the following well known inequality (reminiscent of Heisenberg's uncertainty principle from quantum mechanics) for the uncertainty product:

$$\Delta t \Delta f \geqslant 1 \tag{4}$$

where the equality sign holds if and only if $s(t) = c \exp(-at^2)$ ($a > 0$, $c \in \mathbb{C} \setminus \{0\}$), the Gaussian signal (or *Gabor's elementary signal*), cf Gabor [1], Merzbacher [2].

The inequality in (4) appears to be of primary importance in the study of dolphin echo-location signals. It is known today that dolphins possess a very sophisticated detection and ranging system which is based on the use of very short-duration ultrasound pulses. It has been shown that all of these signals share a remarkably small bandwidth given their time duration. This property of maximum 'concentration' in time and frequency ($\Delta t \Delta f \geqslant 1$) justifies the estimate or approximation of the signal by a parametric waveform model, the elementary Gabor function. This estimate leads in many cases to a representation of the dolphin echo-location signal by a main pulse and a time-delayed replica, the latter due to a reflection inside the animal's head [3].

In order to give an interpretation of the values (significantly greater than unity) found for these signals, Kamminga and Cohen Stuart [3] studied the following Gabor model:

$$s(t) = \exp(-\alpha_1^2 t^2) \exp[j(\omega_0 t + \varphi_1)] + \gamma \exp[-\alpha_2^2 (t - \tau)^2] \exp\{j[\omega(t - \tau) + \varphi_2)]\}. \tag{5}$$

Their plot (p 243, figure 6) shows an interesting phenomenon: if $\Delta t \Delta f$ (UP) is plotted as a function of the time delay τ, for fixed α_1, α_2, ω, φ_1, φ_2, γ then it has local maxima and minima. However, of course, the actual value never drops below the lower bound of unity given in (4), the value attained by a pure Gabor signal (5) with $\gamma = 0$ (cf figure 1 in this Letter, plotting $\Delta t \Delta f$ for (5) with $\alpha_1 = \alpha_2 = 1$, $\varphi_1 = \varphi_2 = 0$, $\omega = 7$ (an arbitrary choice) and γ as indicated).

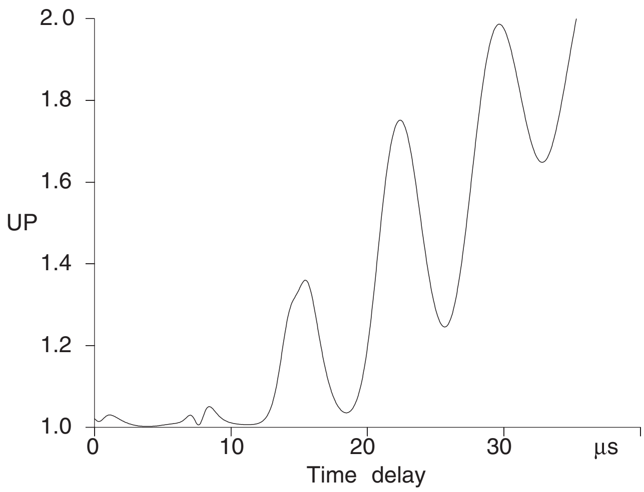

Figure 2. UP for a young *Phocoena phocoena* echo-location signal.

In figure 2, a plot is reproduced based on an actual recorded echo-location signal of a young Harbour porpoise (*Phocoena phocoena*). The dominant frequency is 137 kHz, $\gamma = 0.46$, $\tau = 18 \pm 0.5\,\mu$s. This value of the time delay coincides amazingly well with one of the local minima in the UP plot.

The plots given lead to two questions:

(i) Does this phenomenon also occur in other types of signal?
(ii) If so, is it possible to find a time delay such that the uncertainty product of the composite signal is smaller than that of the pure signal?

Both questions will be answered in the affirmative for the signal that can be given by the derivative of the Gaussian function and for the signal given by the Cauchy distribution. It is conjectured that in general

the uncertainty product of a non-Gaussian composite signal can be smaller than that of the 'pure' signal.

The outline of this Letter is as follows: first, in section 2, the effect of scaling on the uncertainty product will be discussed and then in section 3 the behaviour mentioned in the conjecture will be shown to be true for the derivative of the Gaussian signal, while in section 4 the same will be done for the Cauchy distribution. This is followed by a discussion of some possible implications of this behaviour for the interpretation of models in signal analysis (section 5). Then, in section 6 we reproduce some mathematical formulae to calculate the required quantities and finally a selected list of references is given.

2. Theoretical remarks

Define the 'moments' of the densities by

$$\mu_{t,k} = \int_{-\infty}^{\infty} t^k |s(t)|^2\,\mathrm{d}t \qquad \mu_{f,k} = \int_{-\infty}^{\infty} f^k |\tilde{s}(f)|^2\,\mathrm{d}f \qquad (k \in \mathbb{N}). \qquad (6)$$

The formulae in (2) can then be written as

$$\sigma_t^2 = \frac{\mu_{t,2}}{\mu_{t,0}} - \left(\frac{\mu_{t,1}}{\mu_{t,0}}\right)^2 \qquad \sigma_f^2 = \frac{\mu_{f,2}}{\mu_{f,0}} - \left(\frac{\mu_{f,1}}{\mu_{f,0}}\right)^2. \tag{7}$$

These formulae make it easy to prove the following theorem concerning *scaling invariance*.

Theorem 1. *Let the Fourier pair* $s, \tilde{s}$ *have uncertainty product* $\Delta t \Delta f$; *then the Fourier pair* $s_1, \tilde{s}_1$ *with time domain representation* $s_1(t) = s(\lambda t),\ \lambda > 0$, *satisfies*

$$\Delta_1 t \Delta_1 f = \Delta t \Delta f \tag{8}$$

where the index indicates the values for the scaled functions.

Proof. For s_1 we find

$$\mu_{t,k}^{(1)} = \mu_{t,k}/\lambda^{k+1}$$

and its Fourier transform $\tilde{s}_1(f) = \frac{1}{\lambda}\tilde{s}(\frac{f}{\lambda})$ satisfies

$$\mu_{f,k}^{(1)} = \lambda^{k-1}\mu_{f,k}.$$

Inserting these into (7) and (3) we arrive at (8). □

3. The derivative of the Gauss distribution

Consider the composite signal that is based on the derivative of a Gaussian function:

$$s(t) = t\mathrm{e}^{-t^2}\exp(\mathrm{j}\omega t) + \gamma(t-\tau)\exp\left(-(t-\tau)^2\right)\exp(\mathrm{j}\omega(t-\tau)) \tag{9}$$

where $\omega,\ \tau \in \mathbb{R}$ and $\gamma > 0$.

Using the formulae from section 6 we calculate

$$\mu_{t,0} = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left[1 + 2\gamma(1-\tau^2)\exp(-\tau^2/2)\cos(\omega\tau) + \gamma^2\right]$$

$$\mu_{t,1} = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left\{\gamma\tau[(1-\tau^2)\exp(-\tau^2/2)\cos(\omega\tau) + \gamma]\right\}$$

$$\mu_{t,2} = \frac{1}{16}\sqrt{\frac{\pi}{2}}\left[3 + 2\gamma(3-\tau^4)\exp(-\tau^2/2)\cos(\omega\tau) + (3+4\tau^2)\gamma^2\right].$$

The Fourier transform is given by

$$\tilde{s}(f) = -\pi j\sqrt{\pi}\left[1 + \gamma\exp(-2\pi\mathrm{j}\tau f)\right]\left(f - \frac{\omega}{2\pi}\right)\exp\left\{-\pi^2[f - \omega/(2\pi)]^2\right\}. \tag{10}$$

Again using the formulae from section 6 yields

$$\mu_{f,0} = \frac{1}{4}\sqrt{\frac{\pi}{2}}\left[1 + 2\gamma(1-\tau^2)\exp(-\tau^2/2)\cos(\omega\tau) + \gamma^2\right]$$

$$\mu_{f,1} = \frac{1}{8\pi}\sqrt{\frac{\pi}{2}}\left\{\omega + 2\gamma\exp(-\tau^2/2)[\omega(1-\tau^2)\cos(\omega\tau) - \tau(3-\tau^2)\sin(\omega\tau)] + \omega\gamma^2\right\}$$

$$\mu_{f,2} = \frac{1}{16\pi^2}\sqrt{\frac{\pi}{2}}\big(3 + \omega^2 + 2\gamma\exp(-\tau^2/2)\{[(3-6\tau^2+\tau^4) + \omega^2(1-\tau^2)]\cos(\omega\tau) \\ -2\omega\tau(3-\tau^2)\sin(\omega\tau)\} + (3+\omega^2)\gamma^2\big).$$

From the explicit expressions given above the uncertainty product can be calculated. Figure 3 gives the plot as a function of τ for the example with $\omega = 7$ and the values of γ indicated; the pure signal has value 3 (as it is obvious that we have an even function in τ, only values $\tau \geqslant 0$ have been used).

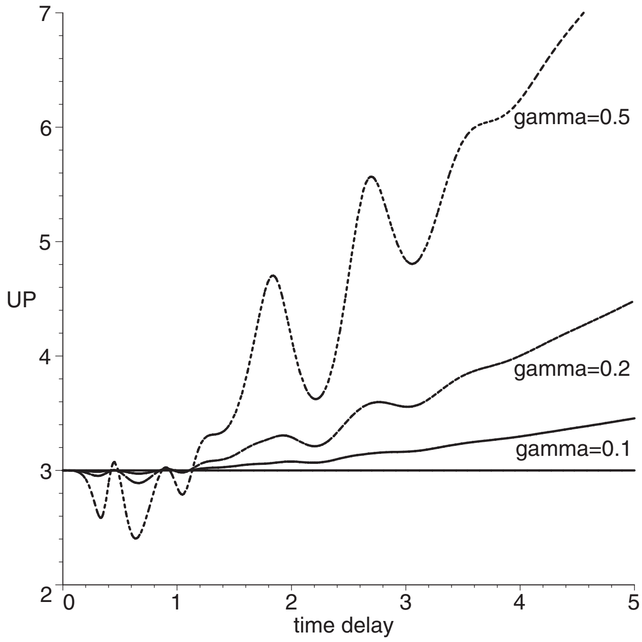

Figure 3. UP for the composite Gauss-derivative signal, $\omega = 7$.

4. The Cauchy distribution

We now turn to the composite signal based on a Cauchy-type signal

$$s(t) = \frac{\exp(\mathrm{j}\omega t)}{1+t^2} + \gamma \frac{\exp[\mathrm{j}\omega(t-\tau)]}{1+(t-\tau)^2} \tag{11}$$

where $\omega,\ \tau \in \mathbb{R}$ and $\gamma > 0$. The moments are

$$\mu_{t,0} = \frac{\pi}{2}\left[1 + \frac{8\gamma\cos(\omega\tau)}{\tau^2+4} + \gamma^2\right]$$

$$\mu_{t,1} = \frac{\pi}{2}\gamma\tau\left[\frac{4\cos(\omega\tau)}{\tau^2+4} + \gamma\right]$$

$$\mu_{t,2} = \frac{\pi}{2}\left[1 + 4\gamma\cos(\omega\tau)\frac{\tau^2+2}{\tau^2+4} + (1+\tau^2)\gamma^2\right].$$

Its Fourier transform is given by

$$\tilde{s}(f) = \pi\left[1+\gamma\exp(-2\pi\mathrm{j}\tau f)\right]\exp\left[-2\pi|f-\omega/(2\pi)|\right] \tag{12}$$

and the moments by

$$\mu_{f,0} = \frac{\pi}{2}\left[1 + \frac{8\gamma\cos(\omega\tau)}{\tau^2+4} + \gamma^2\right]$$

$$\mu_{f,1} = \frac{1}{4}\omega(1+\gamma^2) + 2\gamma\left[\frac{\omega\cos(\omega\tau)}{\tau^2+4} - \frac{2\tau\sin(\omega\tau)}{(\tau^2+4)^2}\right]$$

$$\mu_{f,2} = \frac{1}{2\pi}\left\{\frac{1}{4}(1+\gamma^2)\left(\omega^2+\frac{1}{2}\right) + 2\gamma\left[\frac{\omega^2\cos(\omega\tau)}{\tau^2+4} - \frac{4\omega\tau\sin(\omega\tau)}{(\tau^2+4)^2} + \frac{2(4-3\tau^2)\cos(\omega\tau)}{(\tau^2+4)^3}\right]\right\}.$$

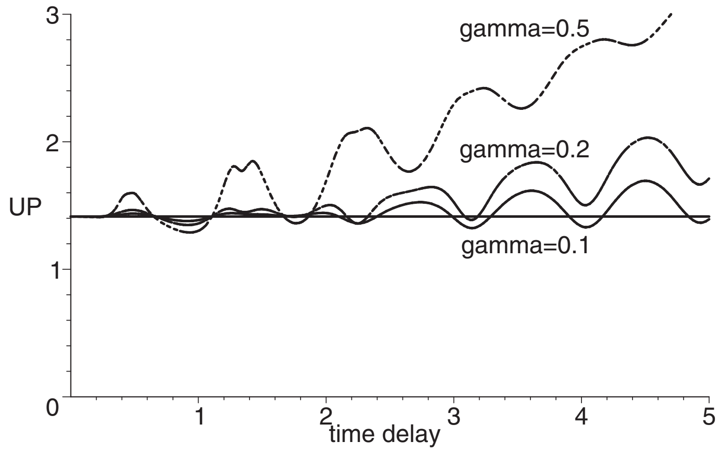

Figure 4. UP for the composite Cauchy signal, $\omega = 7$.

From the explicit expressions given above the uncertainty product can be calculated. Figure 4 gives the plot as a function of τ (as it is obvious that we have an even function in τ, only values $\tau \geqslant 0$ have been used) for the example $\omega = 7$ and the values of γ indicated; the pure signal has value $\sqrt{2}$.

5. Conclusion and discussion

The effect on the uncertainty product of adding to a signal a delayed scaled replica of that signal offers important possibilities for the interpretation of the echo-location sounds emitted by dolphins, especially for dolphins emitting sounds of a polycyclic sonar character (containing more than six complete cycles). Due to the ever-present reflection (originating inside the head of the animal) which is added to the main signal, a UP-value and time delay are found which are surprisingly close to the location of one of the minima in the theoretical plot. Here the animal seems to turn the liability of an ever-persistent reverberation into an asset.

The experimental data imply that not only a main contribution to the signal exists, but also a secondary one *of a similar form*. The peak values of the envelopes of the main (respectively secondary) contribution were found to have a ratio roughly between 3 and 1.5 (cf [3] and [4]).

The examples show that the composite signal can have a smaller uncertainty product than a pure signal of the same form. The data suggest that nature strives to obtain a minimal value for the uncertainty product, even where it does not take the shape of a Gaussian signal, but that of a species-dependent waveform having a time-delayed copy attached to it.

If we calculated the theoretically optimal time delay, and the value of γ and ω, and compared this with the experimentally found distance between the main and secondary peaks, we could give an indication of to what extent the theoretical model of composite signals leads to a better understanding of the process of echo-location used by dolphins and the dependence of the time delay on morphological features of the species.

6. Some formulae

For the sake of completeness the explicit values of integrals used to calculate the uncertainty products in the sections 3 and 4 are given below (cf [5] and [6]).

6.1. Moments

$$\int_{-\infty}^{\infty} t^{2k} \exp(-2t^2)\,\mathrm{d}t = \frac{\Gamma(k+1/2)}{2^{k+1/2}} \qquad (k \in \mathbb{N})$$

$$\int_{-\infty}^{\infty} v^2 \exp(-2v^2)\cos(2\tau v)\,\mathrm{d}v = \frac{1}{4}\sqrt{\frac{\pi}{2}}(1-\tau^2)\exp(-\tau^2/2)$$

$$\int_{-\infty}^{\infty} v^4 \exp(-2v^2)\cos(2\tau v)\,\mathrm{d}v = \frac{1}{16}\sqrt{\frac{\pi}{2}}(3-6\tau^2+\tau^4)\exp(-\tau^2/2)$$

$$\int_{-\infty}^{\infty} v^3 \exp(-2v^2)\sin(2\tau v)\,\mathrm{d}v = \frac{1}{8}\sqrt{\frac{\pi}{2}}\tau(3-\tau^2)\exp(-\tau^2/2)$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(t^2+1)^2} = \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{((t-\tau)^2+1)^2} = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{t\,\mathrm{d}t}{((t-\tau)^2+1)^2} = \frac{\pi\tau}{2}$$

$$\int_{-\infty}^{\infty} \frac{t^2\,\mathrm{d}t}{(t^2+1)^2} = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{t^2\,\mathrm{d}t}{((t-\tau)^2+1)^2} = (1+\tau^2)\frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(t^2+1)((t-\tau)^2+1)} = \frac{2\pi}{\tau^2+4}$$

$$\int_{-\infty}^{\infty} \frac{t\,\mathrm{d}t}{(t^2+1)((t-\tau)^2+1)} = \frac{\pi\tau}{\tau^2+4}$$

$$\int_{-\infty}^{\infty} \frac{t^2\,\mathrm{d}t}{(t^2+1)((t-\tau)^2+1)} = \pi\frac{\tau^2+2}{\tau^2+4}$$

$$\int_{-\infty}^{\infty} \exp(-2|2\pi f-\omega|)\,\mathrm{d}f = \frac{1}{2\pi}$$

$$\int_{-\infty}^{\infty} f\exp(-2|2\pi f-\omega|)\,\mathrm{d}f = \frac{\omega}{4\pi^2}$$

$$\int_{-\infty}^{\infty} f^2\exp(-2|2\pi f-\omega|)\,\mathrm{d}f = \frac{\omega^2+1/2}{8\pi^3}$$

$$\int_{-\infty}^{\infty} \exp(-2|2\pi f-\omega|)\cos(2\pi\tau f)\,\mathrm{d}f = \frac{2\cos(\omega\tau)}{\pi(\tau^2+4)}$$

$$\int_{-\infty}^{\infty} f\exp(-2|2\pi f-\omega|)\cos(2\pi\tau f)\,\mathrm{d}f = \frac{1}{\pi^2}\left[\frac{\omega\cos(\omega\tau)}{\tau^2+4} - \frac{2\tau\sin(\omega\tau)}{(\tau^2+4)^2}\right]$$

$$\int_{-\infty}^{\infty} f^2\exp(-2|2\pi f-\omega|)\cos(2\pi\tau f)\,\mathrm{d}f$$
$$= \frac{1}{2\pi^3}\left[\frac{\omega^2\cos(\omega\tau)}{\tau^2+4} - \frac{4\omega\tau\sin(\omega\tau)}{(\tau^2+4)^2} + \frac{2(4-3\tau^2)\cos(\omega\tau)}{(\tau^2+4)^3}\right].$$

6.2. Fourier transform formulae

$$s(t) = t \exp(-t^2)$$
$$\tilde{s}(f) = -\pi\sqrt{\pi}\mathrm{j}f \exp(-\pi^2 f^2).$$

If $\{s(t),\ \tilde{s}(f)\}$ is a Fourier pair, then also

$$\left\{\exp(\mathrm{j}\omega t)s(t),\ \tilde{s}\left(f - \frac{\omega}{2\pi}\right)\right\}$$

and

$$\{s(t-\tau),\ \exp(-2\pi\mathrm{j}\tau f)\tilde{s}(f)\}.$$

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